Lecture 20 Summary

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1 Structure of an Isolated Vortex

The magnetic field profile $\overrightarrow{h}$ around a vortex in the extreme type-II limit ($\kappa >> 1$) is,

$\nabla^2 \overrightarrow{h} - \frac{1}{\lambda_{eff}^2} \overrightarrow{h} = -\frac{\Phi_0}{\mu_0 \lambda_{eff}^2} \delta_2(r)\widehat{z}$. This equation has an exact solution, the zeroth order Hankel function of complex argument $K_0(x)$,
$\overrightarrow{h}(r) = \frac{\Phi_0}{2\pi\mu_0\lambda_{eff}^2} K_0(r/\lambda_{eff})\widehat{z}$.

This solution has the following asymptotic forms,
For large r, $r > \lambda_{eff}$ it goes as $h(r) \propto r^{-1/2} e^{-r/\lambda_{eff}}$,
and for $\xi_{GL} < r < \lambda_{eff}$ it goes as $h(r) \propto \Big[\log(\frac{\lambda_{eff}}{r}) + 0.12\Big]$.

The class web site shows the full solution, as well as these asymptotic forms, for the field profile as a function of radius. Once you are several times λ_{eff} away from the vortex core, there is essentially no field visible.

The currents can be calculated from the relation $\overrightarrow{\nabla} \times \overrightarrow{h} = \overrightarrow{J}$, and the result is,
$\overrightarrow{J}(r) = \frac{\Phi_0}{2\pi\mu_0\lambda_{eff}^3} K_1(r/\lambda_{eff})\widehat{\theta}$.
The current distribution has the following asymptotic forms,
For large r, $r > \lambda_{eff}$ it goes as $J(r) \propto r^{-1/2} e^{-r/\lambda_{eff}}$,
and for $\xi_{GL} < r < \lambda_{eff}$ it goes as $J(r) \propto 1/r$. This can be re-written as,
$\overrightarrow{J} = |\psi|^2 e^* v_{s\theta}\widehat{\theta}$, with $v_{s\theta} = \frac{\hbar}{m^* r}$. This result has an interesting interpretation. It can be written as $m^* v_{s\theta} r = \hbar$. In other words it resembles the Bohr-Sommerfeld quantization rule that $\oint pdq = n\hbar$, where in this case $n = 1$. In other words the supercurrent around a single vortex carries one unit of angular momentum.
The apparent divergence of the current at small r is cut off by the fact that $v_{s\theta}$ will eventually exceed the critical velocity v_c and the order parameter will be suppressed in the core. One can show that the kinetic energy density in the current flow is comparable to the condensation energy density at the edge of

the core $r \sim \xi_{GL}$.

2 Vortex Energy

We wish to first calculate the energy of a single vortex and then the interaction energy of two vortices. This will also allow us to calculate the force of interaction between two vortices, and finally the force exerted on a vortex by a transport current. All of this will be done in the extreme type-II limit $\kappa >> 1$ in which we effectively ignore the core of the vortex.

The energy per unit length of a vortex can be shown to be $W'_{vortex} = \frac{1}{2\mu_0} \iint_{s_\perp} \overrightarrow{B}(\overrightarrow{r}) \cdot \overrightarrow{V}(\overrightarrow{r}) d^2r$, where $\overrightarrow{B}$ is the magnetic field created by the vortices and $\overrightarrow{V}$ is their vorticity. Since the magnetic field of the single vortex came from solution of a linear equation, we shall assume that the vorticity and magnetic field can be formed from a linear superposition of single-vortex solutions in the case of multi-vortex problems.

For a single vortex, this evaluates to, $W'_{vortex} = \frac{\Phi_0^2}{4\pi\mu_0\lambda_{eff}^2} K_0(\xi_{GL}/\lambda_{eff})$, where the Hankel function is evaluated at the edge of the core because the magnetic field is essentially the same at $r = 0$ as at $r = \xi_{GL}$. In the extreme type-II limit, the argument of K_0 is small, giving $K_0(x) \sim \ln(1/x)$ for $x << 1$. This yields, $W'_{vortex} = \frac{\Phi_0^2}{4\pi\mu_0\lambda_{eff}^2} \ln(\kappa)$.
Note that the energy to create a vortex goes to zero as T approaches T_c, leading to a proliferation of vortex loops. This is one picture for how the superconductor to normal phase transition occurs in three dimensions. In two dimensions it is the first step in the Kosterlitz-Thouless phase transition.

3 Vortex Interactions

Two vortices a distance ℓ apart will have a total energy, $W'_{2vortices} = 2\frac{\Phi_0^2}{4\pi\mu_0\lambda_{eff}^2} K_0(\xi_{GL}/\lambda_{eff}) \pm \frac{\Phi_0^2}{2\pi\mu_0\lambda_{eff}^2} K_0(\ell/\lambda_{eff})$. The first term is twice the self-energy of a vortex, while the second term is the interaction energy of the two vortices. The $\pm$ denotes the case of parallel $(+)$ and anti-parallel $(-)$ vortices. Like vortices repel, while opposites attract.

The interaction force can be deduced from the distance dependence of the interaction energy, $\overrightarrow{f}_{12} = -\partial W'_{2vortices}/\partial \ell = \pm \frac{\Phi_0^2}{2\pi\mu_0\lambda_{eff}^3} K_1(\ell/\lambda_{eff})$. This expression is proportional to the current created at vortex 2 by vortex 1: $\overrightarrow{J}_{12}$. It

can be written as a Lorentz-like force as,
$\overrightarrow{f}_{12} = \overrightarrow{J}_{12} \times \Phi_0 \widehat{z}$